NUMERICAL SIMULATION OF A SYSTEM OF ORDINARY EQUATIONS WITH A SMALL PARAMETER AT THE HIGHEST DERIVATIVE

Nasiba Djurayeva
Department of General Sciences, Surkhandarya Academic Lyceum of the Ministry of Internal Affairs of the Republic of Uzbekistan, Jarkurgan district, Uzbekistan
nasibajt@mail.ru
https://doi.org/10.5281/zenodo.17774360

ARTICLE INFO

Received: 26th November 2025
Accepted: 27th November 2025
Online: 28th November 2025

KEYWORDS

Chebishev polinomials, main flow, two-phase flows, Reynolds number, wave number, relaxation time and mass concentration of particles.

ABSTRACT

Numerical modeling of heterogeneous (multiphase) mixtures or, in particular, gas suspensions (mixtures of gases with solid particles) is one of the important directions in the study of the dynamics of complex hydrodynamic systems. The study of mathematical models of multiphase systems is relevant, primarily from the point of view of applications. Such flows are often encountered in problems of aerodynamics, chemical technology, combustion physics, and others. The study of the influence of suspended particles on the hydrodynamic stability of two-phase flows requires, firstly, a thorough analysis of mathematical models of two-phase media and, secondly, the development of an effective, reliable mathematical apparatus for numerical simulation of hydrodynamic stability equations. In this article, for numerical modeling of the equations of stability of two-phase flows, it is proposed to apply the method of preliminary integration, where Chebyshev polynomials of the first kind are used as basis functions. The essence of the method lies in the fact that before solving the problem, the series in Chebyshev polynomials is pre-integrated the required number of times. Then the integrated series are used in a differential problem and a generalized algebraic eigenvalue problem with complex matrices and vectors is obtained, which is solved by standard methods

Introduction

The use of the spectral and spectral-grid methods for solving the problem of hydrodynamic stability of two-phase flows is described in [1–3]. In these papers, Chebyshev polynomials of the first kind were used as basis functions. Chebyshev polynomials are widely used in the construction of optimal iterative methods, in solving various problems of hydrodynamic stability. Among orthogonal polynomials, only Chebyshev polynomials have a minimax property, which allows more uniform distribution of errors over the entire integration interval. Therefore, all those methods where Chebyshev polynomials are used are optimal or almost optimal. The range of applications of Chebyshev polynomials is increasingly expanding. Let us give a brief overview of numerical methods for the application of Chebyshev polynomials.

Materials and methods

The hydrodynamic stability of plane-parallel flows with suspended particles is studied with respect to small two-dimensional perturbations in the form of traveling waves:

$$\Psi(x,\eta,t)=\psi(\eta)\,\mathrm{e}^{ik(x-\lambda t)}$$

$$\Phi(x,\eta,t)=\varphi(\eta)\,\mathrm{e}^{ik(x-\lambda t)}$$

Here Ψ , Φ are stream functions, and $\psi(\eta)$, $\varphi(\eta)$ are the amplitudes of the stream function, for pure gas and particles, respectively, x and η are longitudinal and transverse coordinates, t is time, k is the wave number, $\lambda=\lambda_r+i\lambda_i$ is an unknown constant to be determined. If $\lambda_i>0$, then the flow under consideration is unstable, if $\lambda_i<0$ - stable. If $\lambda_i=0$, then the oscillations are neutrally stable.

To represent approximate solutions ψ,φ as series in Chebyshev polynomials of the first kind, we map the integration interval $[\eta_0,\eta_\ell]$ onto the interval $[-1,+1]$ using the following change of the independent variable:

$$\eta=\frac{m}{2}+\frac{\overline{\ell}}{2}y,\ y\quad[-1,+1],$$

$$m=\eta_0+\eta_\ell,\ \overline{\ell}=\eta_\ell-\eta_0$$

Thus, we have the following boundary value problem:

$$P\psi+M\varphi=0, \qquad (1)$$

$$S\psi+K\varphi=0\,, \qquad (2)$$

$$\psi(\mathrm{y})=\varphi(y)=\frac{d\psi}{dy}=0\ \text{ at }\ y=-1, \qquad (3)$$

$$\psi(\mathrm{y})=\varphi(y)=\frac{d\psi}{dy}=0\ \text{ at }\ y=1 \qquad (4)$$

where the operators P,M,S,K take the form:

$$P=\frac{1}{i\overline{k}\,\overline{\mathrm{Re}}}\overline{D}^2-(\mathrm{U(y)}-\lambda-\frac{if}{GT})\overline{D}+\frac{d^2U}{dy^2},$$

$$M=\frac{f}{iGT}\overline{D},\ S=\frac{1}{iGT}\overline{D},$$

$$K=-(\mathrm{U(y)}-\lambda-\frac{i}{GT})\overline{D}+\frac{d^2U}{dy^2},$$

$$\overline{D}=\frac{d^2}{dy^2}-\overline{\mathrm{k}}^2,\ \overline{k}=\frac{\ell}{2}k\,,\ \overline{\mathrm{Re}}=\frac{\ell}{2}\mathrm{Re}$$

In the method of preliminary integration, the highest derivatives in differential equations (1) and (2) are represented as the following series:

$$\psi^{(4)}(y)=\sum_{n=0}^{\bar{N}}a_nT_n(y)$$

$$\varphi^{(2)}(y)=\sum_{n=0}^{\bar{N}} d_n T_n(y) \qquad (5)$$

$$\mathrm{U}(y_p^{(j)})=\sum_{n=0}^{\bar{N}} b_n T_n(y_p^{(j)})$$

where

$y_p^{(j)}=\cos(\pi p/\bar{N})$, $p=0,1,\ldots,\bar{N}$

nodes of Chebyshev polynomials, $T_n(y)$ Chebyshev polynomials of the first kind,

$\bar{N}$ - the number of polynomials used to approximate the solution. The expansion coefficients b_n for the main flow in (5) are determined by the following inverse transformation:

$$b_n=\frac{2}{\bar{N}c_n}\sum_{p=0}^{\bar{N}}\frac{1}{c_p}\mathrm{U}(y_p^{(j)})T_n(y_p^{(j)}),\, n=0,1,\ldots,\bar{N}$$

$c_0=c_{\bar{N}}=2$, $c_m=1$ at $m \neq 0,\bar{N}$.

For Chebyshev polynomials, the following recursive formulas are valid:

$$2T=\frac{c_n}{n+1}T_{n+1}^{(1)}-\frac{d_{n-2}}{n-1}T_{n-1}^{(1)},\quad 2yT_n=c_nT_{n+1}+d_{n-1}\,\mathrm{T}_{n-1}, \qquad (6)$$

where $c_n=d_n=0$, if $n<0, c_0=2, d_0=1$,

$c_n=d_n=1$, if $n>0$.

The main advantage of the pre-integration method is that before solving the boundary value problem (1)-(4), series (5) are pre-integrated. Thus, all lower derivatives and desired solutions are found:

$$\psi^{(3)},\psi^{(2)},\psi^{(1)},\psi,\varphi^{(1)},\varphi. \qquad (7)$$

It is known that integration operations improve the smoothness of series (5). At the same time, in spectral methods, series in Chebyshev polynomials are differentiated.

Using preliminary integration, the functions from (7) are determined as follows:

$$\psi^{(3)}=\sum_{j=0}^{\bar{N}+1}\sum_{i=0}^{\bar{N}} f_{ji}^{(3)}a_iT_j+C_1T_0,$$

$$\psi^{(2)}=\sum_{j=0}^{\bar{N}+2}\sum_{i=0}^{\bar{N}} f_{ji}^{(2)}a_iT_j+C_1T_1+C_2T_0,$$

$$\psi^{(1)}=\sum_{j=0}^{\bar{N}+3}\sum_{i=0}^{\bar{N}} f_{ji}^{(1)}a_iT_j+\frac{C_1}{4}T_2+C_1T_1+C_2T_0 \qquad (8)$$

$$\psi=\sum_{j=0}^{\bar{N}+4}\sum_{i=0}^{\bar{N}} f_{ji}^{(0)}a_iT_j+\frac{C_1}{24}T_1+\frac{C_2}{4}T_2+(C_3-\frac{C_1}{8})T_1+C_4T_0,$$

$$\varphi^{(1)}=\sum_{j=0}^{\bar{N}+1}\sum_{i=0}^{\bar{N}} \varphi_{ji}^{(1)}d_iT_j+D_1T_0,$$

$$\varphi=\sum_{j=0}^{\bar{N}+2}\sum_{i=0}^{\bar{N}} \varphi_{ji}^{(0)}d_iT_j+D_1T_1+D_2T_0$$

The constants C_1, C_2, C_3, C_4, D_1, D_2 in formulas (8) are determined from the satisfaction of boundary conditions (3) and (4).

Substituting the values of the constants C_1, C_2, C_3, C_4, D_1, D_2 into (8) we have the following generalizing formulas

$$\psi^{(\beta)}=\sum_{j=0}^{\bar{N}+4-\beta}\sum_{i=0}^{\bar{N}} g_{ji}^{(\beta)}a_iT_j,\beta=0,1,2,3\,,\quad \varphi^{(\beta)}=\sum_{j=0}^{\bar{N}+2-\beta}\sum_{i=0}^{\bar{N}} \bar{g}_{ji}^{(\beta)}d_iT_j,\beta=0,1 \qquad (9)$$

where, for example

$$g_{ji}^{(3)}=f_{ji}^{(3)}+\delta_{j,0}\frac{3}{2}\left(\sigma_i^{(0)}-\delta_i^{(0)}\right)-\left(\sigma_i^{(1)}+\delta_i^{(1)}\right),$$

$$\bar{g}_{ji}^{(1)}=\varphi_{ji}^{(1)}+\delta_{j,0}\frac{3}{2}\left(\varepsilon_i^{(0)}-\Delta_i^{(0)}\right)-\left(\varepsilon_i^{(1)}+\Delta_i^{(1)}\right).$$

other coefficients $g_{ji}^{(\beta)}\left(\beta=0,1,2\right)$ and $\bar{g}_{ji}^{(0)}$ are also determined.

The found coefficients satisfy the considered boundary conditions and their derivatives. We represent the main flow $U(y)$ as the following row

$$U^{(\beta)}(y)=\sum_{n=0}^{N_{b-1+\beta}} b_n^{(\beta)}T_n(y)\,,\ \beta=0,1,2\,, \qquad (10)$$

and different values of parameter β correspond to different velocity profiles of the main oncoming flow. For example, at $\beta=0$ the main flow is the Poiseuille flow:

$U(y)=1-y^2$.

Thus, substituting series (9)-(10) into equations (1), (2) we have a generalized algebraic eigenvalue problem

$$\sum_{k=0}^{\bar{N}}\Bigg(\frac{1}{\bar{i}\bar{k}Re}\left(\delta_{ik}-2\bar{k}^2g_{ik}^{(2)}+k^4g_{ik}^{(0)}\right)-\sum_{i=0}^{N_{b+2}}\sum_{j=0}^{\bar{N}+2}\beta_{i\ell_j}b_i^{(0)}g_{ik}^{(2)}+\bar{k}^2\sum_{i=0}^{N_{b+2}}\sum_{j=0}^{\bar{N}+4}\beta_{i\ell_j}b_i^{(0)}g_{ik}^{(0)}+\sum_{i=0}^{N_b}\sum_{j=0}^{\bar{N}+4}\beta_{i\ell_j}b_i^{(2)}g_{ik}^{(0)}+\frac{\bar{i}f}{GT}\left(g_{ik}^{(2)}-\bar{k}^2g_{ik}^{(0)}\right)\Bigg)a_k+$$
$$\frac{f}{iGT}\sum_{k=0}^{\bar{N}}\left(\bar{g}_{ik}^{(2)}-\bar{k}^2\bar{g}_{ik}^{(0)}\right)d_i=-\lambda\sum_{k=0}^{\bar{N}}\left(g_{ik}^{(2)}-\bar{k}^2g_{ik}^{(0)}\right)a_k,i=0,1,2,...,\bar{N} \qquad (11)$$

$$\sum_{k=0}^{\bar{N}}\frac{1}{\bar{i}GT}\left(\bar{g}_{ik}^{(2)}-\bar{k}^2g_{ik}^{(0)}\right)a_k+\frac{\bar{i}}{GT}\sum_{k=0}^{\bar{N}}\left(\bar{g}_{ik}^{(2)}-\bar{k}^2g_{ik}^{(0)}\right)d_i-\left(\sum_{i=0}^{N_{b+2}}\sum_{j=0}^{\bar{N}+2}\beta_{i\ell_j}b_i^{(0)}\bar{g}_{ik}^{(2)}+\bar{k}^2\sum_{i=0}^{N_{b+2}}\sum_{j=0}^{\bar{N}+4}\beta_{i\ell_j}b_i^{(0)}\bar{g}_{ik}^{(0)}+\sum_{i=0}^{N_b}\sum_{j=0}^{\bar{N}+4}\beta_{i\ell_j}\bar{g}_{ik}^{(2)}\right)d_k=$$
$$=-\lambda\sum_{k=0}^{\bar{N}}\left(\bar{g}_{ik}^{(2)}-\bar{k}^2\bar{g}_{ik}^{(0)}\right)d_k\,,i=0,1,2,...,\bar{N} \qquad (12)$$

where *i* denotes the imaginary unit,

$$\beta_{i\ell_j}=\frac{2}{\pi c_{i-1}}\int_{-1}^{+1}\left(1-y^2\right)^{-1/2}T_iT_\ell T_jdy=\frac{1}{2}\left(\delta_{i,\ell+j}+\frac{1}{c_\ell}\left(\delta_{i,\ell-j}+\delta_{i,j-\ell}\right)\right)$$

here the following identity is used

$2T_jT_\ell=T_{j+\ell}+T_{|j-\ell|}$.

The system of equations (11) and (12) forms an algebraic eigenvalue problem of the following form

$$\left(A-\lambda B\right)X=0 \qquad (13)$$

where A and B are known complex square matrices, λ are eigenvalues of the problem under consideration, X are eigenvectors: moreover, λ and X are complex values:

$\lambda=\left(\lambda_0,\lambda_1,...,\lambda_{2\bar{N}}\right)$

$$X = \begin{matrix} x_{00} & x_{10} & \dots & x_{2\bar{N}+1,0} \\ x_{01} & x_{11} & \dots & x_{2\bar{N}+1,1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{0,2\bar{N}+1} & x_{1,2\bar{N}+1} & \dots & x_{2\bar{N}+1,2\bar{N}+1} \end{matrix}$$

References:

[1] Нормуродов Ч. Б., Соловьев А.С. Устойчивость двухфазных потока газ-твердые частицы в пограничном слое // Изв.РАН.Сер.Механика жидкости и газа.- . –Москва,1987. - №2. – с.60-64

[2] Нормуродов Ч. Б. Математическое моделирование проблемы гидродинамических задач для двухфазных плоскопараллелных течений // Математическое моделирование. — Москва, 2007. №6(19). - с. 53-60.

[3] Абуталиев Ф. Б., Нормуродов Ч. Б. Математическое моделирование проблемы гидродинамической устойчивости. — Т.: Fan va texnologiya, 2011. 188 с.